the motion persists in either a positive or negative precession mode is apparent from the phase plane behavior as characterized by the two spiral-type singularities. The motion in either mode, in general, consists of a superposition of the two coning motions, as described by Murphy's Eq. (7) above. Nevertheless, the distinction is made that the motion is either "positive precession" or "negative precession," depending on whether y is, in general, positive or negative, respectively.

The damping term ν_m was not included in the analysis of Ref. 1. In the absence of Magnus effects, $\nu_m p = 2f\nu p_r$, where $f \equiv C_{N\alpha}'/(C_{N\alpha}' + C_{m_q})$, and the stability criterion can be written

$$\xi = p_r[\dot{p}/p + \nu(1 - 2f)] \tag{6}$$

Therefore, for $C_{N\alpha}' \leq C_{m_q}$ or $f \leq \frac{1}{2}$, ξ is always positive for positive $\dot{p}/p$ and the instability always occurs from the positive to the negative mode. With $C_{N\alpha}' > C_{m_q}$ or with the presence of Magnus effects, the instability could occur in either direction, as Murphy points out. However, for sufficiently large $\dot{p}/p$, ξ can always become positive regardless of the magnitudes of C_{m_q}, $C_{N\alpha}$, and $C_{np\alpha}$. In either case, y must also approach zero for the instability to occur, and this is strongly influenced by roll acceleration.

When the damping term ν_m is included in the angle-of-attack convergence solution derived in Ref. 1, this result takes the form

$$\frac{\bar{\theta}}{\theta_0} = (1 + \sigma^2)^{-1/4} \times \exp\left\{-\frac{1}{2}\int_0^t \left[\frac{\dot{p}}{p} + \nu \pm \frac{\left(\frac{\dot{p}}{p} + \nu - \frac{2}{\mu}\nu_m\right)}{(1 + \sigma^2)^{1/2}}\right] dt'\right\} \tag{7}$$

The effective damping term $\dot{p}/p + \nu$ that first appears under the integral equally damps both precessional motions. However, the relative damping of the two modes is indicated by the numerator of the expression following the $\pm$ sign, which is identically the stability criterion. When $\xi = 0$, this expression is zero and both modes are damped equally. When $\xi > 0$, the positive mode damps more rapidly, indicating a transition from positive mode to negative mode precession and when $\xi < 0$ the transition is in the other direction. Thus, Eq. (7) substantiates the conclusions derived from the phase plane analysis.

The author wishes to thank Murphy for his suggestions that have permitted the more general analysis of the problem presented above.

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Comments on "Temperature Laws for a Turbulent Boundary Layer with Injection and Heat Transfer"

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ISAACSON and AlSaji[1] have derived the expression

$$2u_\tau/Pr_t v_w\ [(1 + v_w t^+/u_\tau)^{1/2} - 1] = A\ \ln(u_\tau y/\nu) + B \tag{1}$$

for the dimensionless temperature t^+ in the inner layer of a transpired boundary layer: Pr_t is the turbulent Prandtl number, A is the reciprocal of the von Karman constant K, and B is in general a function of v_w/u_τ. The results presented by Isaacson and AlSaji, and the further results in Dr. AlSaji's thesis,[2] suggest that B is constant to within experimental error. The formula quoted above was derived from "mixing length" arguments with the further assumption that the ratio of shear stress τ to heat transfer q is independent of y. This last assumption is correct only for $Pr = 1$, $Pr_t = 1$, and it seems worth while to put on record the more accurate form of the "mixing length" temperature profile. The mixing length assumptions are discussed in Ref. 3.

In the inner layer the mean momentum and mean temperature equations reduce to

$$v_w \partial u/\partial y = \partial\tau/\partial y \tag{2}$$

$$v_w \partial T/\partial y = -\partial q/\partial y \tag{3}$$

suppressing density and specific heat, and the mixing length formulae for velocity and temperature are

$$\partial u/\partial y = \tau^{1/2}/Ky \tag{4}$$

$$\partial T/\partial y = -q/\tau^{1/2}K_\Theta y \tag{5}$$

where K/K_Θ is equal to Pr_t by definition of the latter. Isaacson and AlSaji find $K_\Theta = 0.464$; K is about 0.41 (0.424 according to AlSaji).

From Eq. (2–5)

$$\frac{\partial u/dy}{\partial T/\partial y} = \frac{-\partial\tau/\partial y}{\partial q/\partial y} = \frac{-K_\Theta}{K}\frac{\tau}{q} \tag{6}$$

$$\therefore d\tau/dq = K_\Theta\tau/Kq$$

so

$$\frac{q}{q_w} = \left[\frac{\tau}{\tau_w}\right]^{K/K_\Theta} \cdot \text{const}$$

The constant, which represents the effect of the viscous sublayer in which the mixing length formulae are not valid, is unity only if $Pr = 1$, otherwise, the behavior of the heat flux and shear stress in the viscous sublayer will be different. For small v_w/u_τ, $q \to q_w$, and $\tau \to \tau_w$, so the constant is of the form

$$1 + f(v_w/u_\tau) \text{ where } f(0) = 0$$

By an analysis parallel to that of Ref. 1 we obtain

$$t^+ = \frac{u_\tau}{v_w}\left[\left(1 + \frac{u}{u_\tau}\cdot\frac{v_w}{u_\tau}\right)^{Pr_t}(1 + f) - 1\right] \tag{7}$$

so that

$$\frac{2u_\tau}{v_w}\left[\left(\frac{1 + v_w t^+/u_\tau}{1 + f}\right)^{1/2Pr_t} - 1\right] = \frac{1}{K}\ln\frac{u_\tau y}{\nu} + B' \tag{8}$$

where B' is the additive constant in the logarithmic velocity profile, which is in general a function of v_w/u_τ but which seems to be nearly constant for $v_w \geqslant 0$.[2,4–6]

This equation differs from that of Ref. 1 in having $1/Pr_t$ as a factor in the exponent on the left-hand side, and in the presence of the $(1 + f)$ factor. By requiring compatibility with

$$t^+ = (1/K_\Theta)\ln\frac{u_\tau y}{\nu} + B \tag{9}$$

as $v_w/u_\tau \to 0$ we find that f must be

$$(v_w/u_\tau)(B - B') + 0(v_w/u_\tau)^2 \tag{10}$$

near $v_w = 0$, where B' is of course evaluated at $v_w = 0$. The best estimate for $B - B'$ is about -1.1 in air (Pr = 0.7) ac-

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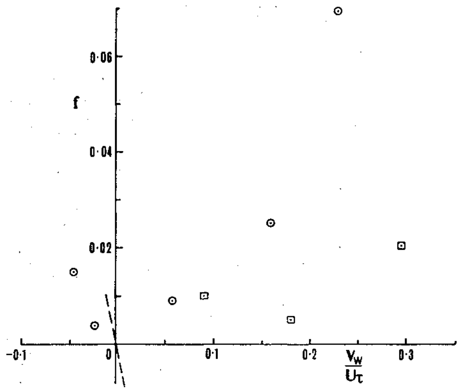

Fig. 1 Variation with injection parameter of the function f in Eq. (8) ⊙; Whitten et al.[7] ⊡; AlSaji[2]. — — —; behavior near origin for compatibility with Eq. (9).

cording to solid-surface data. Isaacson and AlSaji's equation would be correct only if $B = B'$ and $Pr_t = 1$.

Figure 1 shows the behavior of f according to the results of AlSaji[2] and of Whitten et al.[7] In both cases I have taken $Pr_t = 0.91$ which is the best estimate for solid surfaces and agrees well with AlSaji's implied value of 0.424/0.464 = 0.914. I have used the experimental value of u/u_τ at $u_\tau y/\nu = 100$ to find the right-hand side of Eq. (7) because the velocity profile data of Whitten et al. disagree strongly with the finding of constant B' and give $K = K_\Theta = 0.44$: whether this is due to errors in skin-friction measurements or to the effects of surface roughness is not clear, but in either case the best basis for comparison is the measured u/u_τ.

A given percentage difference in $1 + f$ implies a rather larger percentage difference in t^+ so that the differences shown in Fig. 1 are quite significant. Isaacson and AlSaji's Eq. (1), implies $f = 0$, of course, and their results agree with this to within likely experimental error (I have not analyzed all their profiles). The data of Whitten et al. show large variations in f. It is surprising that there is no sign that either set of results fairs smoothly into the required trend of f near the origin: clearly the behavior of f must be near-singular there, indicating that small suction or injection produce comparatively large changes in the heat-transfer properties of the viscous sublayer. This contrasts with the behavior of the velocity profile, for which B' is, if not a constant, at least a smoothly varying function of v_w/u_τ. It would be interesting to see if f remains nearly zero in other fluids; the required slope near the origin varies roughly as $Pr^{3/4}$.

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